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On Non-Excellent Discrete Valutation Rings

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In my Commutative Algebra, I have included (pp.87-88) one of the famous examples of non-catenary noetherian domains due to Nagata. In the second edition of the book I wrote (p.260) that "the ring A of p.88 is a G-ring which is not universally catenary." Recently, Prof. Heinzer of USA pointed out to me that the ring A cannot be a G-ring if the field k is of characteristic p such that $[k : k^p] < \infty$. I checked it and found that he was right. I must have been thinking of the case of characteristic zero only when I wrote that comment of p.260.

Close inspection of the example shows that the problem of whether A is a G-ring or not reduces to knowing whether a certain DVR is a G-ring or not. According to EGA IV, a Noetherian ring A is a Nagata ring (anneau universellement japonais) if and only if the following conditions are satisfied:

(1) for each maximal ideal m of R, the formal fibres of R_m are geometrically reduced; and

(2) for each finite R-algebra S, Nor(S) is open in Spec(S). (When R is local, condition (1) is sufficient.) It follows from this that, if R is one-dimensional, to be a Nagata ring

and to be quasi-excellent are equivalent. On the other hand, all one-dimensional noetherian rings are universally catenary. (For, Ratliff has shown that a noetherian ring R is u.c. if R[X], the polynomial ring in one variable over R, is catenary. If R is one-dimensional then R[X] is two-dimensional, and it is clear from the definition that rings of dimension two or less are catenary.) Therefore one dimensional noetherian rings are excellent if and only if they are Nagata. In particular, when R is a one-dimensional noetherian local domain with completion $\hat{R}$ and quotient field K, the following conditions are all equivalent.

1) R is excellent;

1') R is a G-ring;

2) R is Nagata;

3) for every minimal prime ideal P of $\hat{R}$, the quotient field of $\hat{R}/P$ is separable over K.

Therefore a DVR is excellent if it has characteristic zero, but not necessarily so if it has characteristic $p > 0$. The following example is well known:

<u>COUNTEREXAMPLE</u> 1. Let k be a field of characteristic p such that $[k:k^p] = \infty$. Let $R = k[[X]]$ be the formal power series ring in one variable over k, and consider its subring

$$A = \left\{ z = \sum_0^\infty a_i X^i \in R \;\middle|\; [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}.$$

This ring can also be written $A = (k^p[[X]])[k]$. It is a DVR with prime element X and completion $\hat{A} = R$, and since $A \supset R^p$ it is not excellent.

Up to this point all are well known.

The ring A of Counterexample 1 has the bad property that $[k:k^p] = \infty$. I learned from Heinzer that a counterexample with a perfect residue field could be easily constructed. Although some of you may already know it, I'd like to report it here.

COUNTEREXAMPLE 2. Let k be an arbitrary field of characteristic p. Let z be a formal power series in x with coefficients in k which is transcendental over k(x). (The existnece of such z is shown, e.g. in Zariski-Samuel, Vol.II, p.220.) Set

$$A = k[[x]] \cap k(x,z), \qquad A' = k[[x]] \cap k(x,z^p).$$

Since A and A' contain x, both rings are DVR's with x as prime element; moreover, both have k as residue field. Since A = k + xA = A' + xA, if A were finite over A' we would have A = A' by the Lemma of Nakayama, which is absurd because the quotient field of A is k(x,z) and that of A' is $k(x,z^p)$. Therefore A is not finite over A'. But A is the integral closure of A' in the finite extension k(x, z) of the quotient field $k(x, z^p)$ of A'. This means that A' is a non-excellent DVR with residue field k.

So much I learned from Heinzer. Now, is the ring A in the above example excellent ? If A is finite over A' then a theorem of Greco (Nagoya Math. J. 60(1976)) says that A is a G-ring iff A' is, but in the present case A is not finite over A' and so we must check A separately.

COUNTEREXAMPLE 3. Write the above z as $z = \sum_0^\infty a_i x^i$, and set

$$k' = k^p(a_0, a_1, a_2, \dots).$$

Since the ring $A = k[[x]] \cap k(x,z)$ lies between k[[x]] and $k[x]_{(x)}$, we have $\hat{A} = k[[x]]$.

Case 1. $[k':k] < \infty$. This condition can be also written as $z \in (k^p[[x]])[k]$. Denoting the quotient field of $k[[x]]$ by $k((x))$, we have $(k((x)))^p(x) = k^p((x))$. Set $L = k((x))$. Then $z \in L^p(x,k)$. The field $k(x,z)$ is a rational function field in two variables over k, hence it has the derivation $\partial/\partial z$, which cannot be extended to a derivation of L because $z \in L^p(x,k)$. Therefore L is not separable over $k(x,z)$. (Cf. Th.88 of C.A. 2nd ed.) Hence A is not excellent in this case.

Case 2. $[k':k] = \infty$. In other words $z \notin L^p(k,x)$. Let $C = \{c_\gamma\}_{\gamma\in\Gamma}$ be a p-basis of k (over the prime field). Then it is easy to see that $C \cup \{x,z\}$ is a p-basis of $k(x,z)$. Now, $C \cup \{x,z\}$ is also p-independent in $L = k((x))$. In fact, the p-independence of C in L follows from the fact that any derivation D of k can be extended to a derivation of $k[[x]]$ by

$$D(\Sigma\, a_i x^i) = \Sigma D(a_i)x^i,$$

hence also to a derivation of L. Since $L^p(C) \subset k((x^p))$ we have $x \notin L^p(C)$. Moreover we have $z \notin L^p(C,x) = L^p(k,x)$ by assumption. Therefore $C \cup \{x,z\}$ is p-independent in L. This shows that L is separable over $k(x,z)$, and so A is excellent in this case.

Case 2 does not happen unless $[k:k^p]=\infty$. It is ironical that A is excellent exactly when k and z are 'pathological'. But this phenomenon should be understood in the following sense: the element z in Case 1 is not sufficiently transcendental over $k(x)$.

The counterexample of Nagata starts from the formal power series ring $k[[x]]$ in one variable over a field k. Take an element $z = \Sigma_{i=1}\, a_i x^i$ which is transcendental over $k(x)$, and set $z_1 = z$, $z_2 = a_2x + a_3x^2 + \ldots$, ..., $z_i = a_ix + a_{i+1}x^2 + \ldots$

Then we have

$$(*) \qquad (z_{i+1} + a_i)x = z_i \qquad i = 1,2,\dots .$$

Set $R = k[x,z_1,z_2,\dots] \subset k[[x]]$. The quotient field of R is $k(x,z)$. Set $\mathfrak{m} = xR$. As all z_i are in $\mathfrak{m}$ by (*), we have $R/\mathfrak{m} \simeq k$ and so $\mathfrak{m}$ is a maximal ideal of R. Set also $\mathfrak{n} = (x-1, z)R$. (In C.A.p.88 $\mathfrak{n}$ is defined by $\mathfrak{n} = (x-1,z_1,z_2,\dots)$, but that is a misprint.) Since $R/(x-1) \simeq k[z]$ we have $R/\mathfrak{n} \simeq k$, so that $\mathfrak{n}$ is also a maximal ideal.

Let B denote the localization of R with respect to the multiplicative set $R - (\mathfrak{m} \cup \mathfrak{n})$. Then B is a semi-local domain with maximal ideals $\mathfrak{m}B$ and $\mathfrak{n}B$, and we have $B_{\mathfrak{m}B} \simeq R_{\mathfrak{m}}$, $B_{\mathfrak{n}B} \simeq R_{\mathfrak{n}}$. Since $R_{\mathfrak{m}} \subset k[[x]]$ and $\bigcap_n x^n k[[x]] = (0)$, we have $\bigcap_n x^n R_{\mathfrak{m}} = (0)$ and so $R_{\mathfrak{m}}$ is a DVR. Hence $R_{\mathfrak{m}} = k[[x]] \cap k(x,z)$, which is the DVR of Counterexample 3. As for $R_{\mathfrak{n}}$, since $R \subset k[x,x^{-1},z] \subset R_{\mathfrak{n}}$ we have $R_{\mathfrak{n}} = k[x,x^{-1},z]_P$ where $P = \mathfrak{n}R_{\mathfrak{n}} \cap k[x,x^{-1},z] = (x-1,z)$. Therefore $R_{\mathfrak{n}}$ is an excellent regular local ring of dimension 2. Thus B is a G-ring iff $R_{\mathfrak{m}}$ is excellent.

Set $I = \mathrm{rad}(B) = \mathfrak{m}B \cap \mathfrak{n}B$, $A = k + I$. Then A is a subring of B and I is a maximal ideal of A. Since $B/I \simeq B/\mathfrak{m}B \oplus B/\mathfrak{n}B \simeq k \oplus k$, the ring B is finite over A. This proves that A is a local ring, and is noetherian (Eakin's theorem). Moreover, A is a G-ring iff B is so (by Greco's theorem), that is, iff $R_{\mathfrak{m}}$ is excellen Since $\mathrm{ht}(\mathfrak{n}B) = 2$ and $\mathrm{ht}(\mathfrak{m}B) = 1$, both B and A have dimension 2. Since A is a local domain of dimension 2, it is catenary. But A cannot be universally catenary because the dimnension formula fails to hold between A and B:

$$\mathrm{ht}(\mathfrak{m}B) = 1 \neq \mathrm{ht}(\mathfrak{m}B \quad A) = \mathrm{ht}(I) = 2.$$

End.